INTRODUCTION

Continuum mechanics deals with the relation between forces (stress, σ) and deformation (strain, ε), or deformation rate (strain rate, $\dot{\varepsilon}$). Solid materials, rigid, usually deform elastically, that is the relation between force (stress) and strain is linear, $\varepsilon = \mu\sigma$. Fluids, and viscous materials, that have a linear relation between stress and strain rate, $\dot{\varepsilon} = \eta\sigma$, are called Newtonian fluids and linearly viscous materials, respectively.

Most materials show more complicated behavior, non-linear viscosity, combination of elastic and viscous behavior, and so on.

USEFUL DEFINITIONS

General principles

- **Transformation of axes** A transformation from one orthogonal set of axes to another is given by $x_i^{'} = A_{ij}x_j$.
- **Definition of a tensor** A tensor is a physical quantity whose components transform according to the following laws:
  - Zero-rank tensor (scalar): $a^{'} = a$,
  - First-rank tensor (vector): $p_i^{'} = A_{ij}p_j$,
  - Second-rank tensor: $T_{ij}^{'} = A_{ik}A_{jl}T_{kl}$.
- **Kroneker's delta**

$$\delta_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j. \end{cases}$$

- **The permutation symbol** The permuation symbol ε has the following properties,

$$\varepsilon_{ijk} = \begin{cases} 0, \text{if two, or three, of the integers } i, j, k \text{ are equal,} \\ 1, \text{if } (ijk) \text{ is an even permutation of (123),} \\ -1, \text{if } (ijk) \text{ is an odd permutation of (123).} \end{cases}$$

- **Summation convention** Repeated indices indicate summation, for example

$$a = n_i n_i = n_1 n_1 + n_2 n_2 + n_3 n_3.$$

Products

$$\begin{aligned}
\text{Dot product:} \quad & a = \mathbf{v} \cdot \mathbf{k} && \Leftrightarrow a = v_m k_m, \\
\text{Tensor-vector relation:} \quad & \mathbf{v} = \mathbf{A} \cdot \mathbf{p} && \Leftrightarrow v_i = A_{ij} p_j, \\
\text{Contraction:} \quad & a = \mathbf{C} : \mathbf{B} && \Leftrightarrow a = C_{kl} B_{kl}, \\
\text{Outer product:} \quad & \mathbf{A} = \mathbf{b} \otimes \mathbf{n} && \Leftrightarrow A_{ij} = b_i n_j.
\end{aligned}$$

STRESS

Stress, $\sigma = F/A$, is one of the fundamental variables of continuum mechanics. The units of stress are force per area, or Pascal, [N m-2 = Pa].

Body and surface forces

The forces acting on a parcel of material are of two types:

- Body forces. Affect the whole volume. An example is the force of gravity,
- Surface forces. Act on the surface of the material. Friction is an example.

In continuum mechanics surface forces, stress, which is force per area, play an important role.

Definition of stress

Consider a force $\mathbf{t}(\mathbf{n})$, acting on a surface with a unit normal $\mathbf{n}$ (Figure 1a). Considering a small parcel of the material the stress vectors have equal and opposite signs on the opposite sides (Figure 1b). The stress vector can be resolved into a component normal to the surface, extension, and parallel to the surface, shear (Figure 1c).

Stress tensor

Stress is a tensor quantity,

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}. \tag{1}$$

The first indices refers to the direction the force is acting, and the second to the surface the force is action on, i.e. σ_{xz} refers to the stress acting in direction $\mathbf{x}$ on surface $\mathbf{z}$.

Of the 9 components, only 6 are independent (when there is no acceleration), since $\sigma_{ij} = \sigma_{ji}$. If that wasn't true, the little parcel in Figure 1 would rotate.

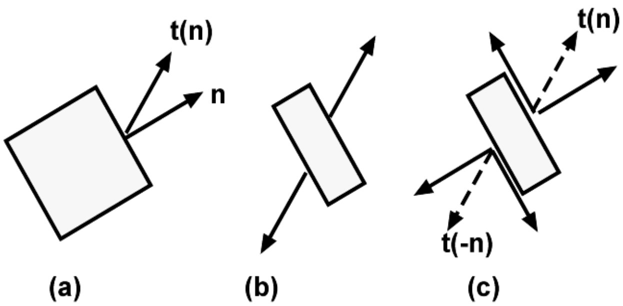

Figure 1: (a) Relation between unit surface normal $\mathbf{n}$, and the stress vector $\mathbf{t}(\mathbf{n})$. (b) The stress vectors on opposite sides of the parcel are equal in magnitude. (c) Stress vectors (dashed arrows) are resolved into components of shear (parallel to the surface) and extension (normal to the surface).

Principal stress

Principal stress and the associated directions play an important role in seismology, for instance. The principal stress components are the eignvalues of the stress tensor. They are usually written as $\sigma_1 > \sigma_2 > \sigma_3$. In the coordinate system defined by the eigenvectors there are no shear stresses. In two dimensions this can be visualized as rotating the coordinate system until the shear stress is zero. If we call the rotated coordinate system x'y' and the angle between the axes θ, then we get:

$$\tau_{x'y'} = \frac{1}{2}(\sigma_{yy} - \sigma_{xx}) \sin 2\theta + \tau_{xy} \cos 2\theta. \tag{2}$$

(T+S 2-32) When $\tau_{x'y'} = 0$ we get

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}. \tag{3}$$

The principal stress is then,

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\frac{(\sigma_{xx} - \sigma_{yy})^2}{4} + \tau_{xy}^2}. \tag{4}$$

Specific stress states

- Hydrostatic pressure: All principal stresses are equal, $p = \sigma_1 = \sigma_2 = \sigma_3$. An example is the stat of stress in stationary fluids,
- Lithostatic pressure: All principal stress components equal to the load of the overlying rock,
- Mean pressure: $p = (\sigma_1 + \sigma_2 + \sigma_3)/3$,
- Deviatoric stress: $\sigma'_{ij} = \sigma_{ij} - p\delta_{ij}$, for example $\sigma'_{xx} = \sigma_{xx} - p$, and $\sigma'_{xz} = \sigma_{xz}$.

STRAIN

Strain refers to the change in shape of a material. Consider a deformable string, such as shown in Figure 2. The string is attached to the origin, and two arbitrary points, one at x and the other at $x+\Delta x$, a distance Δx away from each other, are marked on the string in the un-stretched state (Fig. 2a). After deformation the points are at $x+u$ and $x+u+\Delta x+\Delta u$, where the length of the element is now $\Delta x+\Delta u$ (Fig. 2b). The strain is then,

$$\frac{\text{change in length}}{\text{original length}} = \frac{\Delta x + \Delta u - \Delta x}{\Delta x} = \frac{\Delta u}{\Delta x}. \tag{5}$$

The strain at point P is defined as

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}. \tag{6}$$

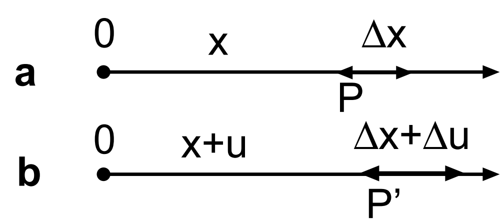

Figure 2: Deformable string, (a) before deformation, and (b) after stretching.